UNIT-5 MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

MY REVISION TIMELINE:-

SUMMARY:-

- A rigid body is the one in which the distances between different particles remain constant.
- For regular shaped bodies with uniform mass distribution, centre of mass always lies at the geometrical centre.
- Net torque produces turning motion in rigid body.
- A rigid body is in translational equilibrium if the total external force on it is zero. It is in rotational equilibrium if the total external torque on it is zero.
- The centre of gravity of an extended body is that point where the total gravitational torque on the body is zero.
- If the external torque acting on the body is zero, the component of angular momentum along the axis of rotation is constant.
- There are rotational equivalents for all the translational quantities.
- Rolling motion is the combination of translational and rotational motions.
- Rolling can also be treated as the momentary rotation about the point of contact.
- In pure rolling, the total kinetic energies is the sum of kinetic energies of rotational and translational motion.
- In sliding the translational motion is more than rotational motion.
- In slipping the rotational motion is more than translational motion.

FORMULAS:-

- Position of centre of mass in vector form

$$\overrightarrow{r_{CM}} = \frac{\sum m_i \vec{r_i}}{M}$$

- Centre of mass of two masses

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

- Torque $\vec{\tau} = \vec{r} \times \vec{F}$
- $\vec{\tau} = (rFsin\theta)\hat{n}$ [Magnitude]
- Torque and angular acceleration $\vec{\tau} = I\vec{\alpha}$
- Magnitude of angular momentum L = rpsinθ
- Angular momentum $\vec{L} = \vec{r} \times \vec{p}$
- Angular momentum and angular velocity $\vec{L} = I\vec{\omega}$
- Torque and angular momentum $\tau = \frac{dL}{dt}$
- Principle of moments $d_1F_1 = d_2F_2 \Rightarrow \frac{F_1}{F_2} = \frac{d_2}{d_1}$

- Bending of cyclist in curves $\theta = \tan^{-1}(\frac{v^2}{rg})$
- Moment of inertia of a uniform rod

  $I = \frac{M}{l}\int x^2 dx \Rightarrow I = \frac{1}{12}Ml^2$
- Moment of inertia of a uniform ring

  $I = MR^2$
- Moment of inertia of a uniform disk

  $I = \frac{1}{2}MR^2$
- Radius of gyration (K)

  $I = MK^2$

  $K = \sqrt{\frac{r_1^2+r_2^2+r_3^2+\cdots+r_n^2}{n}}$
- Parallel axis theorem

  $I = I_C + Md^2$
- Perpendicular axis theorem

  $I_Z = I_X + I_Y$
- Work done by torque, $dW = \tau d\theta$
- Kinetic energy in rotational motion = KE = $\frac{1}{2}I\omega^2$
- Work done by force, dw = Fds
- Kinetic energy in translational motion = KE = $\frac{1}{2}Mv^2$
- Rotational kinetic energy and angular momentum

  $KE = \frac{L^2}{2I}$
- Power delivered by torque $P = \bar{\tau}\omega$
- Kinetic energy in pure rolling = KE = $\frac{1}{2}Mv_{CM}^2[1 + \frac{K^2}{R^2}]$
- Rolling on an inclined plane

  Acceleration

  $a = \frac{g\sin\theta}{[1+\frac{K^2}{R^2}]}$

  Velocity

  $v = \sqrt{\frac{2gh}{[1+\frac{K^2}{R^2}]}}$

  Time taken

  $t = \sqrt{\frac{2h[1+\frac{K^2}{R^2}]}{g\ \sin^2\theta}}$

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MULTIPLE CHOICE QUESTIONS:-

1. **The centre of mass of a system of particles does not depend upon**
   d) force acting on particle
2. **A couple produces**
   a) pure rotation
3. **A particle is moving with a constant velocity along a line parallel to positive x-axis. The magnitude of its angular momentum with respect to the origin is**
   d) remaining constant
4. **A rope is wound around a hollow cylinder of mass 3kg and radius 40cm. What is the angular acceleration of the cylinder if the rope is pulled with a force 30N?**
   b) 25 rad s^{-2}
   Solution:
   m = 3kg
   r = 40 × 10^{-2} m = 0.4m
   F = 30 N
   Moment of inertia of hollow cylinder
   $I = MR^2$
   $I = 3\ (0.4)^2$
   $I = 0.48 kgm^2$
   $FR = I\alpha$
   $30 \times 0.4 = 0.48\alpha$
   $\alpha = \frac{12}{0.48} = \frac{1200}{48}$
   $\boldsymbol{\alpha = 25\ rad\ s^{-2}}$
5. **A closed cylinder container is partially filled with water. As the container rotates in a horizontal plane about a perpendicular bisector, its moment of inertia**
   a) increases
6. **A rigid body rotates with an angular momentum L. If its kinetic energy is halved, the angular momentum becomes**
   d) $\frac{L}{\sqrt{2}}$
   Solution:
   $KE = \frac{1}{2} I\omega^2$; $L = I\omega$; $KE = \frac{L^2}{2^2}$
   $\therefore KE\ \alpha\ L^2 \Rightarrow E_1\ \alpha\ L_1^2$ and $E_2\ \alpha\ L_2^2$
   $\frac{E_1}{E_2} = (\frac{L_1}{L_2})^2$
   Here $E_1 = E$, $E_2 = \frac{E}{2}$, $L_1 = L$, $L = ?$
   $\frac{E}{\frac{E}{2}} = (\frac{L}{L_2})^2$
   $\frac{2E}{E} = (\frac{L}{L_2})^2$

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$$\frac{L}{L_2} = \sqrt{2}$$

$$\boxed{L_2 = \frac{L}{\sqrt{2}}}$$

7. **A particle undergoes uniform circular motion. The angular momentum of the particle remain conserved about**
   a) the centre point of the circle
8. **When a mass is rotating in a plane about a fixed point, its angular momentum is directed along**
   a) a line perpendicular to the plane of rotation
9. **Two discs of same moment of inertia rotating about their regular axis passing through center and perpendicular to the plane of disc with angular velocities ω_1 and ω_2. They are brought into contact face to face coinciding the axis of rotation. The expression for loss of energy during this process is**
   a) $\frac{1}{4} I (\omega_1 - \omega_2)^2$
   Solution:
   Moment of inertia of a disc passing through centre $I = \frac{1}{2}MR^2$
   Energy of the first disc $= \frac{1}{4} I\omega_1^2$
   Energy of the second disc $= \frac{1}{4} I\omega_2^2$
   $$\boxed{\textit{Loss of energy} = \frac{1}{4} I (\omega_1 - \omega_2)^2}$$
10. **A disc of moment of inertia I_a is rotating in a horizontal plane about its symmetry axis with a constant angular speed ω. Another disc initially at rest of moment of inertia I_b is dropped coaxially on to the rotating disc. Then, both the disc rotate with same constant angular speed. The loss of kinetic energy due to friction in this process is**
    d) $\frac{1}{2}\frac{I_b I_b}{(I_a + I_b)}\omega^2$
    Solution:
    The moments of inertia of two discs are I_a and I_b respectively. The angular velocity of the disc A is ω.
    The sum of kinetic energies of two discs before coming in contact is
    $k_1 = \frac{1}{2} I_a \omega_1^2 + \frac{1}{2} I_b \omega_2^2$
    But the angular velocity of the disc b is $\omega_2 = 0$ *(rest)*
    ⇨ $k_1 = \frac{1}{2} I_a \omega_1^2$
    The final kinetic energy of the two disc system
    $k_2 = \frac{1}{2}\frac{I_a^2 \omega_1^2}{I_a + I_b}$

The loss of kinetic energy $k_1 - k_2 = \frac{1}{2} I_a \omega_1^2 - \frac{1}{2} \frac{I_a^2 \omega_1^2}{I_a + I_b}$

$$= \frac{1}{2} \frac{[I_a(I_a+I_b)\omega^2 - I_a^2\omega^2]}{I_a+I_b}$$

$$\boxed{= \frac{1}{2} \frac{I_b I_b}{(I_a+I_b)} \omega^2}$$

11. The ratio of the acceleration for a solid sphere(mass m and radius R) rolling down an incline of angle θ without slipping and sliding down the incline without rolling is

a) 5:7

Solution:

Acceleration of the solid sphere while rolling down without slipping

$a_1 = \frac{g \sin\theta}{1 + \frac{K^2}{r^2}}$

Acceleration developed while slipping down

$a_2 = g \sin\theta$

Required ratio $\frac{a_1}{a_2} = \frac{\frac{g \sin\theta}{1+\frac{K^2}{r^2}}}{g \sin\theta}$

$$\frac{a_1}{a_2} = \frac{1}{1+\frac{K^2}{r^2}}$$

For solid sphere $\frac{K^2}{r^2} = \frac{2}{5}$

Ratio of acceleration $\frac{a_1}{a_2} = \frac{1}{1+\frac{2}{5}} = \frac{1}{\frac{5+2}{5}} = \frac{5}{7}$

$$\boxed{\therefore a_1 : a_2 = 5 : 7}$$

12. From a disc of radius R, a mass M, a circular hole of diameter R whose rim passes through the center is cut. What is the moment of inertia of the remaining axis passing through it

b) $\frac{13MR^2}{32}$

Solution:

Moment of inertia of a disc $I_1 = \frac{MR^2}{2}$

Mass of small disc $= \frac{M}{\pi R^2} \times \pi \times (\frac{R}{2})^2 = \frac{M}{\pi R^2} \times \frac{\pi R^2}{4} = \frac{M}{4}$

By the theorem of parallel axis, the moment of inertia of small disc about an axis passing through o is

$I_2 = \frac{1}{2} \times \frac{M}{4} (\frac{R}{2})^2 + \frac{M}{4} (\frac{R}{2})^2$

$I_2 = \frac{M}{8} \times \frac{R^2}{4} + \frac{M}{4} \times \frac{R^2}{4}$

$I_2 = \frac{MR^2}{32} + \frac{MR^2}{16}$

$I_2 = \frac{MR^2 + 2MR^2}{32}$

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$I_2 = \frac{3MR^2}{32}$

$I = I_1 - I_2$

$I = \frac{MR^2}{2} - \frac{3MR^2}{32}$

$I = \frac{16MR^2 - 3MR^2}{32}$

$$\boldsymbol{I = \frac{13MR^2}{32}}$$

13. The speed of a solid sphere after rolling down from rest without sliding on an inclined plane of vertical height h is

b) $\boldsymbol{\sqrt{\frac{10}{7}gh}}$

Solution:

Potential energy = Translational kinetic energy + Rotational kinetic energy

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{2}{5}MR^2 \times \frac{v^2}{R^2} \quad (\omega = \frac{v}{R})$

$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$

$mgh = \frac{7mv^2}{10}$

$v^2 = \frac{10gh}{7}$

$$\boldsymbol{v = \sqrt{\frac{10gh}{7}}}$$

14. The speed of the center of a wheel rolling on a horizontal surface is v_o. A point on the rim in level with the center will be moving at a speed of

c) $\boldsymbol{\sqrt{2}v_0}$

Solution:

$v_0 = r\omega$; $v_0 \propto r$

For a wheel(uniform ring) the distance of a point on the rim in level with the center [i.e radius] is $\sqrt{2}r$

∴The speed of the center is $\sqrt{2}v_0$.

15. A round object of mass M and radius R rolls down without slipping along an inclined plane. The frictional force

d) converts translational energy into rotational energy.

NUMERICAL PROBLEMS:-

1. Given:

Mass of uniform disc M = 100g = 100 × 10^{-3}kg

Diameter d = 10cm

Radius r = 5cm = 5 × 10^{-2}m

v = 20cms^{-1} = 20 × 10^{-2}ms^{-1}

Total energy = ?

Formula used:

When rolling, the total kinetic energy of the disc = Translational KE + Rolling KE

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\frac{v^2}{R^2} \quad \left[I = \frac{1}{2}mr^2 \text{ and } \omega = \frac{v}{R}\right]$$

$$E = \frac{1}{2}mv^2 + \frac{1}{4}mv^2$$

$$E = \frac{3}{4}mv^2$$

Solution:

$$E = \frac{3}{4}mv^2$$

$$E = \frac{3}{4} \times 100 \times 10^{-3}(20 \times 10^{-2})^2$$

$$E = \frac{3}{4} \times 100 \times 10^{-3} \times 400 \times 10^{-4}$$

E = 30000 × 10^{-7} J or 0.003J

2. Given:

Mass of a particle = 5units

Uniform speed v = 3$\sqrt{2}$ units

y = x+4

Magnitude of Angular momentum = ?

Formula used:

Angular momentum $L = I\omega$

$L = mr^2\omega$

$L = mvr \left(\omega = \frac{v}{r}\right)$

Solution:

$L = mvr$

$L = 5 \times 3\sqrt{2} \times 2\sqrt{2}$

L = 60 units

3. Given:

Initial angular velocity ω_0 = 20π rad s^{-1}

Final angular velocity ω = 40π rad s^{-1}

Time = 10s

Number of rotations in that period = ?

Formula used:

Angular acceleration $\alpha = \frac{\omega - \omega_0}{t}$

Angular displacement $\theta = \omega_0 t + \frac{1}{2}dt^2$

Number of rotations in that period n = $\frac{\theta}{2\pi}$

Solution:

$$\alpha = \frac{\omega - \omega_0}{t}$$

$$\alpha = \frac{40\pi - 20\pi}{10}$$

$$\alpha = \frac{20\pi}{10}$$

$$\alpha = 2\pi$$

$$\alpha = 2 \times 3.14$$

$$\alpha = 6.28\, rad/s^2$$

$$\theta = \omega_0 t + \frac{1}{2} dt^2$$

$$\theta = 20\pi \times 10 + \frac{1}{2} \times 2\pi \times 10^2$$

$$\theta = 200\pi + 100\pi$$

$$\theta = 300\pi\ rad$$

Number of rotations in that period n = $\frac{\theta}{2\pi} = \frac{300\pi}{2\pi}$ *= 150*

n = 150

4. Given:

Mass = m

Length = l

Angle of rotation = θ

Moment of inertia about this axis = ?

Formula used:

Moment of inertia of a uniform rod about one end $I_0 = \frac{ml^2}{12}$

But this axis makes an angle θ *with an axis of rotation* $I_0 = \frac{ml^2 \sin^2\theta}{12}$

Solution:

Moment of inertia = $I_0 + \frac{ml^2 \sin^2\theta}{4}$

Moment of inertia = $\frac{ml^2 \sin^2\theta}{12} + \frac{ml^2 \sin^2\theta}{12}$

Moment of inertia = $\frac{ml^2 \sin^2\theta}{3}$

5. Given:

Mass of one particle = 1kg

Mass of another particle = 3kg

Velocity at the centre of mass of the system = ?

Formula used:

Centre of mass of the system = $\frac{m_1x_1 + m_2x_2}{m_1 + m_2}$

Solution:

Centre of mass of the system = $\frac{m_1x_1+m_2x_2}{m_1+m_2}$

Centre of mass of the system = $\frac{1x_1+3x_2}{1+3}$

Centre of mass of the system = $\frac{1x_1+3x_2}{4}$

Under mutual attraction, when they move towards each other the velocity of their center of mass is ***zero.***

6. Given:

Mass of each hydrogen atom = 1.7 × 10^{-27} kg

Inter atomic distance = 4 × 10^{-10}m

Distance of each hydrogen atom from the axis of rotation = 2 × 10^{-10}m

Moment of inertia of a hydrogen molecule = ?

Formula used:

Moment of inertia I = mr^2

Solution:

Moment of inertia of a hydrogen molecule:

$I = mr^2 + mr^2$

$I = 2mr^2$

$I = 2 \times 1.7 \times 10^{-27} \times (2\times10^{-10})^2$

$I = 2 \times 1.7 \times 4 \times 10^{-47}$

$I = 13.6 \times 10^{-47}$

$$\boldsymbol{I = 1.36 \times 10^{-46} kgm^2}$$

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