UNIT-10 OSCILLATIONS

MY REVISION TIMELINE:-

SUMMARY:-

- When an object or a particle moves back and forth repeatedly about a reference point for some duration of time it is said to have oscillating or vibrating motion.
- For a SHM, the acceleration or force on the particle is directly proportional to its displacement from a fixed point and always directed towards that fixed point. The force is
  $F_x = -kx$
  Where k is a constant whose dimension is force per unit length called as force constant.
- In Simple harmonic motion, the displacement, $y = A\sin\omega t$
- In Simple harmonic motion, the velocity, $v = A\omega\cos\omega t = \omega\sqrt{A^2 - y^2}$
- In Simple harmonic motion, the acceleration, $a = \frac{d^2y}{dt^2} = -\omega^2 y$
- The time period is defined as the time taken by a particle to complete one oscillation. It is usually denoted by T. Time period $T = \frac{2\pi}{\omega}$
- The number of oscillations produced by the particle per second is called frequency. It is denoted by f. SI unit for frequency is s^{-1} or hertz. Mathematically, frequency is related to time period by $f = \frac{1}{T}$
- The frequency of the angular harmonic motion is $f = \frac{1}{2\pi}\sqrt{\frac{K}{I}}Hz$
- For n spring connected in series, the effective spring constant in series is $\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots + \frac{1}{k_n} = \sum_{i=0}^{n}\frac{1}{k_i}$
- For n spring connected in parallel, the effective spring constant in parallel is $k_p = \sum_{i=1}^{n} k_j$
- The time period for U-Tube oscillation is $T = 2\pi\sqrt{\frac{1}{2g}}second$
- For a conservation system in one dimension, the force field can be derived from a scalar potential energy $F = -\frac{du}{dx}$
- In a Simple harmonic motion, the potential energy is $U(x) = \frac{1}{2}m\omega^2x^2$
- In a Simple harmonic motion, the kinetic energy is $KE = \frac{1}{2}mv_x^2 = \frac{1}{2}m\omega^2(A^2 - x^2)$
- Total energy for a Simple harmonic motion is $E = \frac{1}{2}m\omega^2A^2$ = constant
- Types of oscillation: Free oscillation, Damped oscillation, Maintained oscillation and Forced oscillation.
- Resonance is a special case of forced oscillation.

One two academy

FORMULAS:-

- $F_x = -kx$
- $x = +A\sin\omega t$
- $y = A\sin\omega t$
- $a = \frac{d^2y}{dt^2} = -\omega^2 y$
- $v = A\omega\cos\omega t = \omega\sqrt{A^2 - y^2}$
- $T = \frac{2\pi}{\omega}$
- $f = \frac{1}{T}$
- $f = \frac{1}{2\pi}\sqrt{\frac{K}{I}}Hz$
- The effective spring constant in series is $\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \cdots + \frac{1}{k_n} = \sum_{i=0}^{n}\frac{1}{k_i}$
- The effective spring constant in parallel is $k_p = \sum_{i=1}^{n} k_j$
- U-Tube oscillation is $T = 2\pi\sqrt{\frac{1}{2g}}second$
- For a conservation system in one dimension $F = -\frac{du}{dx}$
- In a Simple harmonic motion, the potential energy is $U(x) = \frac{1}{2}m\omega^2x^2$
- In a Simple harmonic motion, the kinetic energy is $KE = \frac{1}{2}mv_x^2 = \frac{1}{2}m\omega^2(A^2 - x^2)$
- Total energy for a Simple harmonic motion is $E = \frac{1}{2}m\omega^2A^2$ = constant

MULTIPLE CHOICE QUESTIONS:-

1. **In a simple harmonic oscillation, the acceleration against displacement for one complete oscillation will be**
   a) a straight line
2. **A particle executing SHM crosses points A and B with the same velocity. Having taken 3s in passing from A to B, it returns to B after another 3s. The time period is**
   c) 12s
   Solution:
   Time period is the time taken by particle to return to B = 4×3

   = 12s

3. **The length of a second's pendulum on the surface of the Earth is 0.9 m. The length of the same pendulum on the surface of planet X such that the acceleration of the planet X is n times greater than the Earth**
   a) 0.9n
   Solution:
   $l = 0.9m$
   $T = 2\pi\frac{l}{g}$
   $T' = 2\pi\frac{l_x}{g_x}$
   $g_x = ng$

$\therefore x = 0.9n$

4. A simple pendulum is suspended from the roof of a school bus which moves in a horizontal direction with an acceleration a, then the time period is

b) T $\alpha \frac{1}{\sqrt{g^2+a^2}}$

Solution:

$T = 2\pi\sqrt{\frac{l}{g}}$

When a bus is moving $g' \sqrt{g^2 + a^2}$

$\therefore T \alpha \frac{1}{\sqrt{g^2+a^2}}$

5. Two bodies A and B whose masses are in the ratio 1:2 are suspended from two separate massless springs of force constants k_A and k_B respectively. If the two bodies oscillate vertically such that their maximum velocities are in the ratio 1 : 2, the ratio of the amplitude A to that of B is

b) $\sqrt{\frac{K_B}{8K_A}}$

Solution:

$v_A : v_B = 1:2$

$A_A : A_B = \sqrt{K_B} : \sqrt{8K_A}$

$\frac{A_A}{A_B} = \sqrt{\frac{K_B}{8K_A}}$

6. A spring is connected to a mass m suspended from it and its time period for vertical oscillation is T. The spring is now cut into two equal halves and the same mass is suspended from one of the halves. The period of vertical oscillation is

b) $T' = \frac{T}{\sqrt{2}}$

7. Two bodies A and B whose masses are in the ratio 1:2 are suspended from two separate massless springs of force constants k_A and k_B respectively. If the two bodies oscillate vertically such that their maximum velocities are in the ratio 1 : 2, the ratio of the amplitude A to that of B is

a) $T = 4\pi\sqrt{m\left(\frac{1}{k_1}+\frac{1}{k_2}\right)}$

Solution:

$T = 2\pi\sqrt{\frac{m}{k}}$

$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

$T' = 2\pi\sqrt{m\left(\frac{1}{k_1}+\frac{1}{k_2}\right)}$

One two academy

8. A simple pendulum has a time period T_1. When its point of suspension is moved vertically upwards according to as $y = kt^2$, where y is vertical distance covered and k = 1 ms^{-2}, its time period becomes T_2. Then, $\frac{T_1^2}{T_2^2}$ is ($g = 10ms^{-2}$)

c) 6/5

Solution:

$T = T_1 = 2\pi\sqrt{\frac{l}{g}}$

$T_1^2 = 4\pi^2\frac{l}{g}$

$y = kt^2 \qquad k = 1ms^{-2}$

$y = t^2 \qquad y \propto t^2$

$\frac{y_1}{y_2} = \frac{t_1^2}{t_2^2}$

$$\boxed{\frac{T_1^2}{T_2^2} = \frac{y_1}{y_2} = \frac{6}{5}}$$

9. An ideal spring of spring constant k, is suspended from the ceiling of a room and a block of mass M is fastened to its lower end. If the block is released when the spring is un-stretched, then the maximum extension in the spring is

c) $2\frac{Mg}{k}$

Solution:

$F = -kx$

$\therefore x = \left|\frac{-F}{k'}\right| = \frac{F}{k'}$

$F = Mg$ and $k' = \frac{k}{2}$

$\therefore x = \frac{Mg}{\frac{k}{2}}$

$$\boxed{x = 2\frac{Mg}{k}}$$

10. A pendulum is hung in a very high building oscillates to and fro motion freely like a simple harmonic oscillator. If the acceleration of the bob is 16 ms^{-2} at a distance of 4m from the mean position, then the time period is

d) πs

Solution:

$a = 16ms^{-2} \qquad y = 4m$

$T = 2\pi\sqrt{\frac{l}{g}}$

$a = \frac{-g}{l} \qquad x = -\omega^2 x$

$16 = -\omega^2 4$

$\omega^2 = \frac{16}{4} = 4$

$\omega = 2$

Time period $T = \frac{2\pi}{\omega}$

$T = \frac{2\pi}{2}$

$\boxed{T = \pi s}$

11. A hollow sphere is filled with water. It is hung by a long thread. As the water flows out of a hole at the bottom, the period of oscillation will

a) first increases and then decreases

12. The damping force on an oscillator is directly proportional to the velocity. The units of the constant of proportionality are

c) kg s^{-1}

Solution:

$F_d \; \alpha \; v$

$F_d = -bv$

$F_d = kv$

$k = \frac{F_d}{v}$

Unit of $k = \frac{kg\ ms^{-1}}{ms^{-1}}$

$\boxed{k = kg\ s^{-1}}$

13. When a damped harmonic oscillator completes 100 oscillations, its amplitude is reduced to $\frac{1}{3}$ of its initial value. What will be its amplitude when it completes 200 oscillations?

d) $\frac{1}{9}$

Solution:

Amplitude at any instant is given by $a = a_o\, e^{-bt}$

a_o *= initial amplitude*

b *= damping constant*

Case 1:

$t = 100T$

$a = \frac{a_o}{3}$

$\frac{a_o}{3} = a_o\, e^{-b100T}$

$e^{-b100T} = \frac{1}{3}$ → *(1)*

Case 2:

$a = a_o\, e^{-b200T}$

$a = a_o\, e^{(-100bT)2} = a_o \left[\frac{1}{3}\right]^2$

$a = \frac{a_o}{9}$

One two academy

14. Which of the following differential equations represents a damped harmonic oscillator?

b) $\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + y = 0$

Solution:

For a damped oscillation F α v

Total restoring force F = -ky – bv

If F = -ky – bv then $\frac{md^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$

$b \rightarrow$ *damping constant*

$y \rightarrow$ *displacement*

$m \rightarrow$ *mass*

We get $\frac{d^2y}{dt^2} + \frac{b}{m} + \frac{dy}{dt} + \frac{ky}{m} = 0$

If k=m and $\frac{b}{m} = \gamma$

Then the equation is

$$\boxed{\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + y = 0}$$

15. If the inertial mass and gravitational mass of the simple pendulum of length l are not equal, then the time period of the simple pendulum is

a) $T = 2\pi\sqrt{\frac{m_i l}{m_g g}}$

Solution:

$T = 2\pi\sqrt{\frac{m}{k}}$

Inertial mass $M_1 \alpha m_i l$

Gravitational mass $M_2 \alpha m_g l$

$\frac{T_1}{T_2}\alpha\sqrt{\frac{M_1}{M_1}}\alpha\sqrt{\frac{m_i l}{m_g g}}$

$$\boxed{T = 2\pi\sqrt{\frac{m_i l}{m_g g}}}$$

NUMERICAL PROBLEMS:-

1. Solution:

Mass of the Earth = M

Radius of the Earth = R

Suppose a body of mass m is dropped into the depth below the surface = d

Acceleration due to gravity $g' = g\left[1 - \frac{d}{R}\right] = g\left[\frac{R-d}{R}\right]$

If Y is the distance of the particle from the centre of the earth, then R-d = Y

$g' = g\frac{Y}{R}$

Force acting on the particle is

$F = -mg'$

$F = \frac{-mgY}{R}$

$F \alpha Y$

$F = mg$

$Y = R$

Force constant $k = \frac{F}{Y} = \frac{mg}{R}$

Time period of oscillation of the particle is $T = 2\pi\sqrt{\frac{m}{k}}$

$T = 2\pi\sqrt{\frac{m}{\frac{mg}{R}}}$

$$\boxed{T = 2\pi\sqrt{\frac{R}{g}}}$$

2. Given:

Length of the pendulum = 0.9m

Angle of inclination = 45°

Formula used:

The effective value of acceleration due to gravity(g) will be equal to the component of g normal to the inclined plane which is $g' = g\cos\theta$

$T = 2\pi\sqrt{\frac{l}{g'}}$

$T = 2\pi\sqrt{\frac{l}{g\cos\theta}}$

Solution:

$g' = g\cos\theta$

$g' = 9.8\cos 45°$

$g' = \frac{9.8}{1.414}$

$g' = 6.9306\ m/s^2$

$T = 2\pi\sqrt{\frac{l}{g\cos\theta}}$

$T = 2(3.14)\sqrt{\frac{0.9}{6.9306}}$

$T = 6.28\sqrt{0.129858}$

$$\boxed{T = 2.262\ second}$$

3. Solution:

When a piece of wood is pressed and released

$F = ma$, m = volume × density

Change in force $= mg = A \times \rho g$

Acceleration $a = \frac{F}{m}$

one two academy

$$a = \left[\frac{A\rho g}{m}\right] x \rightarrow (1)$$

For SHM, $a = \omega^2 x \rightarrow (2)$

From equation 1 and 2

$$\omega^2 = \frac{A\rho g}{m}$$

$$\textbf{Time period } T = 2\pi\sqrt{\frac{m}{A\rho g}}$$

4. Given:

$x = A\sin(\omega t - \varphi) \rightarrow (1)$

$y = B\sin\omega t \rightarrow (2)$

Formula used:

sin(A-B) = sinA cosB + cosA sinB

Solution:

$x = A\sin\omega t\cos\varphi + A\cos\omega t\sin\varphi$

$x - A\sin\omega t\cos\varphi = A\cos\omega t\sin\varphi$

Squaring on both sides

$(x - A\sin\omega t\cos\varphi)^2 = (A\cos\omega t\sin\varphi)^2 \rightarrow (3)$

From (2) $\frac{Y}{B} = \sin\omega t$

$\Rightarrow \cos^2\omega t = 1 - \sin^2\omega t = 1 - \frac{Y^2}{B^2}$

Equation 3 becomes

$(x - A\frac{Y}{B}\cos\varphi)^2 = A^2\left[1 - \frac{Y^2}{B^2}\right]\sin^2\varphi$

On expansion we get $\frac{A^2Y^2}{B^2}\cos^2\varphi - \frac{2xAY}{B}\cos\varphi = A^2\sin^2\varphi - \frac{A^2Y^2}{B^2}\sin^2\varphi$

$$x^2\frac{A^2Y^2}{B^2}(\sin^2\varphi + \cos^2\varphi) - \frac{2xAY}{B}\cos\varphi = A^2\sin^2\varphi$$

Divided by A^2*, we get*

$$\frac{x^2}{A^2} + \frac{Y^2}{B^2} + 1 - \frac{2xY}{AB}\cos\varphi = \sin^2\varphi \rightarrow (4)$$

Hence proved

Special cases:

a) $\varphi = 0$ *in equation 4, we get*

$$\frac{x^2}{A^2} + \frac{Y^2}{B^2} + 1 - \frac{2xY}{AB} = 0 \text{ (or) } \left[\frac{X}{A} - \frac{Y}{B}\right]^2 = 0$$

$$\frac{X}{A} = \frac{Y}{B}$$

$$Y = \frac{B}{A}X$$

The above equation resembles equation of straight line passing through the origin with the slope.

b) $\varphi = \pi$ *in equation 4, we get*

$$\frac{x^2}{A^2} + \frac{Y^2}{B^2} + \frac{2xY}{AB} = 0$$

one two academy

$\left[\frac{X}{A}+\frac{Y}{B}\right]^2 = 0$

$\frac{X}{A} = -\frac{Y}{B}$

$Y = \frac{-B}{A}X$

The above equation resembles equation of straight line passing through the origin with the negative slope.

c) $\varphi = \frac{\pi}{2}$ *in equation 4, we get*

$\frac{x^2}{A^2}+\frac{Y^2}{B^2} = 1$

The above equation is an equation of ellipse with centre at origin.

d) $\varphi = \frac{\pi}{2}$ *and A=B in equation 4, we get*

$\frac{x^2}{A^2}+\frac{Y^2}{A^2} = 1$

$x^2 + Y^2 = A^2$

e) $\varphi = \frac{\pi}{4}$ *in equation 4, we get*

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\frac{x^2}{A^2}+\frac{Y^2}{B^2}-\frac{2xY}{AB} = \frac{1}{2}$

$\frac{x^2}{A^2}+\frac{Y^2}{B^2}-\frac{2xY}{AB}\frac{1}{\sqrt{2}} = \frac{1}{2}$

The above equation is an ellipse.

5. Solution:

a) *Suppose a particle of mass m executes SHM of time period T. The displacement of the particle of any instant A is given by*

$y = A\sin\omega t$

velocity v $= \frac{dy}{dt} = \frac{d}{dt}(A\sin\omega t) = \omega A\cos\omega t$

Kinetic energy $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2\omega t$

Potential energy $E_p = \frac{1}{2}m\omega^2y^2 = \frac{1}{2}m\omega^2A^2\sin^2\omega t$

Average kinetic energy one period of oscillation is

$E_{p_{av}} = \frac{1}{T}\int_0^T E_p dt$

$E_{p_{av}} = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2A^2\sin^2\omega t\, dt$

$E_{p_{av}} = \frac{1}{2T}m\omega^2A^2\int_0^T\left[\frac{1-\cos 2\omega t}{2}\right]dt$

$E_{p_{av}} = \frac{1}{4T}m\omega^2A^2\left[t-\frac{\sin 2\omega t}{2\omega}\right]_0^T$

$E_{p_{av}} = \frac{1}{4T}m\omega^2A^2$

Clearly $E_{k_{av}} = E_{p_{av}}$

Average value of kinetic energy = average value of potential energy

b) *Total energy TE =* $\frac{1}{2}m\omega^2y^2+\frac{1}{2}m\omega^2(A^2-y^2)$

But y = A sinωt

$TE = \frac{1}{2}m\omega^2A^2\omega t + \frac{1}{2}m\omega^2A^2cos^2\omega t$

$TE = \frac{1}{2}m\omega^2A^2(sin^2\omega t + cos^2\omega t)$

But $sin^2\omega t + cos^2\omega t = 1$

$TE = \frac{1}{2}m\omega^2A^2$

Average potential energy = Average kinetic energy = ½ (Total energy)

6. Solution:

Case a)

Pulley is fixed rigidly here. When the mass is displaced by y, the spring will also stretch by y. Hence

F = T = ky

$T = 2\pi\sqrt{\frac{m}{k}}$

Case b)

Mass displace by y, the pulley also displaces by y. Hence

T = 4ky

$T = 2\pi\sqrt{\frac{m}{k}}$

One two academy